

Measure-transmission metric and stability of structured population models

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Abstract

In [Gwiazda, Jamróz, Marciniak-Czochra 2012] a framework for studying cell differentiation processes based on measure-valued solutions of transport equations was introduced. Under application of the so-called measure-transmission conditions it enabled to describe processes involving both discrete and continuous transitions. This framework, however, admits solutions which lack continuity with respect to initial data. In this paper, we modify the framework from [Gwiazda, Jamróz, Marciniak-Czochra 2012] by replacing the flat metric, known also as bounded Lipschitz distance, by a new Wasserstein-type metric. We prove, that the new metric provides stability of solutions with respect to perturbations of initial data while preserving their continuity in time. The stability result is important for numerical applications.

Keywords: transport equation, measure-valued solutions, metrics on measures, structured population models, cell differentiation, stability

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1 Introduction

Cell differentiation process is a biological phenomenon, in which immature cells of living organisms give rise to more mature, i.e. more specialized, ones, see e.g. [1]. In humans, this process takes place primarily during gestation, childhood and adolescence. During these initial stages of human development a fertilized egg cell, called zygote, divides and differentiates multiple times, giving eventually rise to mature cells of blood, muscles, skin, brain etc. In some tissues, the process of cell differentiation persists during adulthood.

For instance, neural stem cells or neural progenitors, which reside in the part of brain called hippocampus, can differentiate (Fig. 1) to become eventually mature neurons, which has implications for human memory, see e.g. [2, 3].

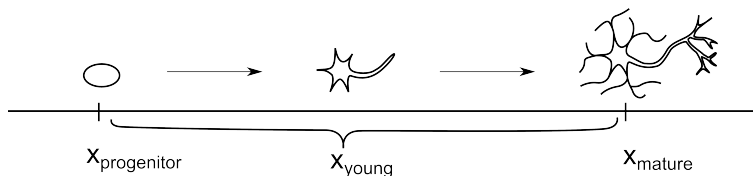


Figure 1: Schematic drawing of process of differentiation of neurons in hippocampus. From the discrete state of neural progenitor a cell differentiates to become a young neuron. This continuous phase lasts around four weeks and consists in migration and morphological maturation. Finally, the young neuron reaches the discrete state of maturity.

Various mathematical models, focusing on different aspects of the process of cell differentiation, and using various mathematical structures, have been proposed in scientific literature. They include modeling differentiation switches via Markov chains or systems of ordinary differential equations (see [4, 5, 6]), modeling the inherent stochasticity via branching processes (see e.g. [7, 8, 9]), modeling delays via delay differential equations (see [10, 11, 12] and references therein), modeling spatial dynamics via discrete lattice models or reaction-diffusion equations (see [13, 14]) and others.

The approach developed in the present paper is called *structured population models*. It consists in tracing populations of cells according to their *maturity level* which is described by a real structure variable $x \in \mathbb{R}$. The order on states x is inherited from \mathbb{R} , which means that state x_2 is *more differentiated* (i.e. more specialized, more mature) than state x_1 iff $x_1 < x_2$. This, in turn, means that a cell from state x_1 can differentiate into a cell in state x_2 yet not vice versa. We distinguish two types of states:

- *discrete states*, in which cells can stay for a positive period of time (e.g. state of stem cell, state of mature cell),
- *continuous states*, which cells pass without halting (e.g. the group of states corresponding to maturing neuron).

Depending on the topology of the state space we distinguish three basic groups of structured population models of cell differentiation:

- discrete models, with state space being a finite subset of \mathbb{R} and composed of discrete states only; the dynamics is based on systems of ODEs, see e.g. [15, 16, 17, 18],
- continuous models, with state space being an interval and composed of continuous states only; the evolution of population of cells is then described by a time-dependent density $u(t, x)$ or, more generally, time-dependent positive Radon measure $\mu(t) \in \mathcal{M}(\mathbb{R})$ which evolves according to the transport (balance) equation $\partial_t \mu + \partial_x(g\mu) = p\mu$, see [19, 20, 21, 22, 23],
- mixed models, which have both discrete and continuous parts, see [24].

In [25] continuous and mixed models of cell differentiation were embedded into a general framework based on measure-valued solutions of transport equations. We refer to this paper for motivations and further biological background as well as derivation of constituents of the model. Mathematically, framework from [25] reads as follows:

$$\partial_t \mu(t) + \partial_x(g_1(v(t)) \mathbf{1}_{x \neq x_i}(x) \mu(t)) = p(v(t), x) \mu(t), \quad (1.1)$$

$$g_1(v(t)) \frac{D\mu(t)}{D\mathcal{L}^1}(x_i^+) = c_i(v(t)) \int_{\{x_i\}} d\mu(t), \quad i = 0, \dots, N \quad (1.2)$$

$$\mu(0) = \mu_0, \quad (1.3)$$

where $t \in \mathbb{R}^+$ and $x \in \mathbb{R}$. $x_0 < x_1 < \dots < x_N$ is a finite collection of points in \mathbb{R} , which correspond to discrete states. $\mathbf{1}_{x \neq x_i}$ is equal 1 if $x \in (x_0, x_1) \cup (x_1, x_2) \cup \dots \cup (x_{N-1}, x_N)$ and 0 otherwise. $\frac{D\mu}{D\mathcal{L}^1}$ denotes the density of measure μ with respect to the one-dimensional Lebesgue measure and $v(t) := \int_{\{x_N\}} d\mu(t)$ denotes the mass of point x_N . The initial datum μ_0 is a Radon measure supported on the interval $[x_0, x_N]$.

Under certain assumptions on coefficients (see [25, Assumptions 3.2]) it was proven that there exists a unique solution

$$\mu \in C([0, \infty), (\mathcal{M}, \rho_F))$$

of problem (1.1)-(1.3). Here, $\mathcal{M} = \mathcal{M}(\mathbb{R})$ is the space of nonnegative Radon measures on \mathbb{R} (see [26] for an introduction to measure theory) and $C([0, \infty), (\mathcal{M}, \rho_F))$ is the space of continuous functions on $[0, \infty)$

with values in space \mathcal{M} equipped with the flat metric ρ_F , which is an adaptation of Wasserstein metric used in the theory of optimal transport, see [27]. This metric, known also under the name bounded Lipschitz distance, is defined by

$$\rho_F(\mu_1, \mu_2) := \sup_{\psi \in Lip^b(\mathbb{R}), |\psi| \leq 1, Lip(\psi) \leq 1} \int_{\mathbb{R}} \psi d(\mu_1 - \mu_2), \quad (1.4)$$

where $Lip^b(\mathbb{R})$ is the set of bounded Lipschitz continuous functions on \mathbb{R} and $Lip(\psi)$ is the Lipschitz constant of ψ .

The starting point for the present research is the fact that the space $C([0, \infty), (\mathcal{M}, \rho_F))$ is incompatible with the structure of problem (1.1)-(1.3) in the sense highlighted by the following example.

Example 1 (Instability in flat metric). *Take $N = 2$ and let $g_1 \equiv 1$ and $c_1 \equiv 0$ in (1.1)-(1.3). For initial condition $\mu_0 = \delta_{x_1}$ the unique solution of problem (1.1)-(1.3) in the sense of [25, Definition 3.3] is given by*

$$\mu(t) = \delta_{x_1}(dx).$$

Here, $\delta_{x_1}(dx)$ denotes a Dirac mass concentrated in x_1 .

For a perturbed initial condition $\mu_0^\varepsilon = \delta_{x_1+\varepsilon}$, on the other hand, we have

$$\mu^\varepsilon(t) = \delta_{x_1+\varepsilon+t}(dx).$$

Using formula (1.4), we obtain $\rho_F(\mu(t), \mu^\varepsilon(t)) = t + \varepsilon$. This means that

- $\rho_F(\mu_0, \mu_0^\varepsilon) = \varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$,
- $\rho_F(\mu(t), \mu^\varepsilon(t)) = t + \varepsilon \rightarrow t$ as $\varepsilon \rightarrow 0$.

Hence, solutions are neither continuous nor stable with respect to initial data.

The goal of the present paper is to introduce a new metric, ρ_{MT} , which better reflects the structure of system (1.1)-(1.3) and admits a stability result, which we subsequently prove.

The paper is organised as follows. In Section 2 we introduce a new metric on Radon measures and discuss its properties. In Section 3 we present the modified framework of cell differentiation and state the main stability theorem. Section 4 is devoted to its proof and discussion. Finally, in Appendix we gather additional estimates used in the proofs.

2 Metrics on the space of measures and measure-transmission metric

In this section, we study a general class of metrics on Radon measures on \mathbb{R} . We discuss and motivate the selection of the one appropriate for system (1.1)-(1.3) – the measure transmission metric ρ_{MT} .

Definition 2 (General class of metrics on $\mathcal{M}(\mathbb{R})$). *Let μ_1, μ_2 be two finite Radon measures on \mathbb{R} . Define*

$$\rho(\mu_1, \mu_2) := \sup_{\psi \in TFS} \int_{\mathbb{R}} \psi d(\mu_1 - \mu_2), \quad (2.1)$$

where TFS (Test Function Space) is a given subspace of $\mathcal{B}(\mathbb{R})$ (Borel functions on \mathbb{R}).

The most important examples of metrics and their TFSs are summarized in Table 1.

Proposition 3. *Formula (2.1) defines a metric provided that TFS satisfies:*

- i) *If $\psi \in TFS$ then $-\psi \in TFS$,*

Name of metric	Test Function Space (TFS)	Notation
Norm (strong) distance	$\{\psi \in \mathcal{B}(\mathbb{R}) : \sup \psi \leq 1\}$	$\ \cdot\ $
Measure-Transmission metric	Defined below	ρ_{MT}
1-Wasserstein distance	$\{\psi \in \text{Lip}(\mathbb{R}) : \text{Lip}(\psi) \leq 1\}$	ρ_W
Bounded Lipschitz distance or flat metric	$\{\psi \in \text{Lip}(\mathbb{R}) : \text{Lip}(\psi) \leq 1, \sup \psi \leq 1\}$	ρ_F

Table 1: Metrics on the space of Radon measures and their Test Function Spaces.

ii) The set $\{af : f \in TFS, 0 < a < \infty\}$ contains all smooth compactly supported functions.

Proof. By assumption i)

$$\rho(\mu_1, \mu_2) = \rho(\mu_2, \mu_1).$$

Next, if μ_1, μ_2, μ_3 are finite Radon measures then

$$\int_{\mathbb{R}} \psi d(\mu_1 - \mu_3) = \int_{\mathbb{R}} \psi d(\mu_1 - \mu_2) + \int_{\mathbb{R}} \psi d(\mu_2 - \mu_3).$$

Taking the supremum over $\psi \in TFS$, we obtain

$$\rho(\mu_1, \mu_3) \leq \rho(\mu_1, \mu_2) + \rho(\mu_2, \mu_3).$$

Finally, suppose that $\mu_1 \neq \mu_2$. Then $\sigma = \mu_1 - \mu_2$ is a signed measure. From the Hahn-Jordan decomposition theorem (see e.g. [28, Theorem 4.1.4 and Corollary 4.1.5]) we obtain positive Radon measures σ^+, σ^- and disjoint Borel sets N, P such that $\sigma^+(N) = 0$, $\sigma^-(P) = 0$ and $\sigma = \sigma^+ - \sigma^-$. Since $\sigma \neq 0$, $\sigma^+(P) > 0$ or $\sigma^-(N) > 0$. Without loss of generality, assume that $\sigma^+(P) > 0$. Then there exists a ball $B(0, R)$ such that

$$\sigma^+(P \cap B(0, R)) > 0.$$

Take $\psi = \mathbf{1}_{P \cap B(0, R)}$ and $\psi^\varepsilon = \psi * \rho^\varepsilon$, where ρ^ε is the standard mollifier. We have

$$\int_{\mathbb{R}} \psi^\varepsilon d(\mu_1 - \mu_2) = \int_{\mathbb{R}} \psi^\varepsilon d\sigma^+ - \int_{\mathbb{R}} \psi^\varepsilon d\sigma^-.$$

Using the fact that ψ^ε is bounded by 1 for every $\varepsilon > 0$ and $\psi^\varepsilon \rightarrow \psi$ pointwise, we pass to the limit in all the terms and obtain

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}} \psi^\varepsilon d(\mu_1 - \mu_2) = \int_{\mathbb{R}} \psi d\sigma^+ - \int_{\mathbb{R}} \psi d\sigma^- = \int_{P \cap B(0, R)} d\sigma^+ > 0.$$

Hence, for ε small enough we have $\int_{\mathbb{R}} \psi^\varepsilon d(\mu_1 - \mu_2) > 0$, which means that $\rho(\mu_1, \mu_2) > 0$. \square

Corollary 4. *Norm distance, 1-Wassertein distance and bounded Lipschitz distance are metrics on $\mathcal{M}(\mathbb{R})$.*

The choice of metric, equivalent to the choice of TFS, is dictated by properties of the system that is being modelled. In case of physical or biological models $\rho(\mu_1, \mu_2)$ should reflect the energy necessary to transform system represented by μ_1 into system represented by μ_2 . Large value of $\rho(\mu_1, \mu_2)$ means that transformation from μ_1 to μ_2 is energetically expensive. Conversely, small value of $\rho(\mu_1, \mu_2)$ means that configurations μ_1 and μ_2 are energetically close to each other. Let us consider a generic example.

Example 5. *Let $\mu_1 = \delta_0$ and $\mu_2 = \delta_\varepsilon$, where $0 < \varepsilon \ll 1$. Then*

$$\int_{\mathbb{R}} \psi d(\mu_1 - \mu_2) = \psi(0) - \psi(\varepsilon).$$

Taking $\psi(x) = \mathbf{1}_{(-\infty, 0]}(x) - \mathbf{1}_{(0, \infty)}(x)$, where $\mathbf{1}_A(x)$ equals 1 if $x \in A$ and 0 otherwise, we obtain that $\|\mu_1 - \mu_2\| = 2$. On the other hand,

$$\rho_F(\mu_1, \mu_2) = \rho_W(\mu_1, \mu_2) = \varepsilon,$$

which follows by observing that $\text{Lip}(\psi) \leq 1$ implies $\psi(0) - \psi(\varepsilon) \leq \varepsilon$ and taking test function

$$\psi(x) = \mathbf{1}_{(-\infty, 0]}(x) + (1 - x)\mathbf{1}_{(0, 2)}(x) + (-1)\mathbf{1}_{[2, \infty)}(x).$$

Example 5 shows that in $\|\cdot\|$ every pair of different states x is distant from one another. Contrarily, in ρ_F and ρ_W the distance of states represented by close enough points x_1 and x_2 is equal to $|x_1 - x_2|$.

Measure-Transmission metric

The *Measure-Transmission metric* ρ_{MT} on $\mathcal{M}(\mathbb{R})$ is a combination of flat metric and norm distance. It is well adapted to cell differentiation models, which are considered in this paper.

To motivate its choice, let $x_0 < x_1 < \dots < x_N$ be points in \mathbb{R} , which correspond to discrete states of system (1.1)-(1.3). We demand $\rho_{MT}(\delta_{x_i}, \delta_{x_i+\varepsilon})$ to be large for $0 < \varepsilon \ll 1$ and $\rho_{MT}(\delta_{x_i}, \delta_{x_i-\varepsilon})$ to be small for $0 < \varepsilon \ll 1$. This can be obtained by taking a TFS, which is composed of functions which are Lipschitz-continuous on intervals $(x_{i-1}, x_i]$, see Figure 2.

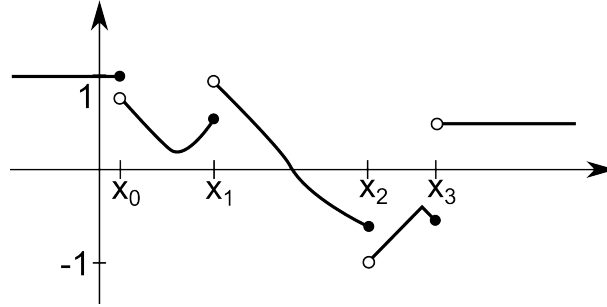


Figure 2: Exemplary test function belonging to the space $B_{MT}(\mathbb{R})$ of test functions for the measure-transmission metric. The function is bounded by 1 and Lipschitz-continuous with constant 1 on intervals $(x_{i-1}, x_i]$.

The space, the norm in it and the unit ball are defined as follows.

Definition 6 (Test function space for ρ_{MT}). *Let $x_0 < x_1 < \dots < x_N$ be arbitrary points in \mathbb{R} . We define:*

$$\begin{aligned} W_{MT}^b(\mathbb{R}) &:= \left\{ \psi \in \mathcal{B}(\mathbb{R}) : \sup |\psi| < \infty, \|\psi|_{(-\infty, x_0]}\|_{\text{Lip}} < \infty, \|\psi|_{(x_0, x_1]}\|_{\text{Lip}} < \infty, \dots, \right. \\ &\quad \left. \|\psi|_{(x_{N-1}, x_N]}\|_{\text{Lip}} < \infty, \|\psi|_{(x_N, +\infty)}\|_{\text{Lip}} < \infty \right\}. \\ \|\psi\|_{W_{MT}^b} &:= \max \left(\sup |\psi|, \|\psi|_{(-\infty, x_0]}\|_{\text{Lip}}, \|\psi|_{(x_0, x_1]}\|_{\text{Lip}}, \dots, \|\psi|_{(x_{N-1}, x_N]}\|_{\text{Lip}}, \|\psi|_{(x_N, +\infty)}\|_{\text{Lip}} \right). \\ B_{MT}(\mathbb{R}) &:= \{ \psi \in W_{MT}^b : \|\psi\|_{W_{MT}^b} \leq 1 \}. \end{aligned}$$

W_{MT}^b equipped with norm $\|\cdot\|_{W_{MT}^b}$ is a Banach space as a direct product of a finite number of Banach spaces of Lipschitz continuous functions on $(x_{i-1}, x_i]$ for $i \in \{0, \dots, N+1\}$, where $x_{-1} := -\infty, x_{N+1} := \infty$.

Definition 7 (Measure-transmission metric). *Let μ_1, μ_2 be finite Radon measures on \mathbb{R} . We define the measure-transmission metric by*

$$\rho_{MT}(\mu_1, \mu_2) := \sup_{\psi \in B_{MT}(\mathbb{R})} \int_{\mathbb{R}} \psi d(\mu_1 - \mu_2).$$

Proposition 8. ρ_{MT} is a metric.

Proof. Follows by Proposition 3. □

Example 9. $\rho_{MT}(\delta_{x_1}, \delta_{x_1+\varepsilon}) = 2$, whereas $\rho_{MT}(\delta_{x_1}, \delta_{x_1-\varepsilon}) = \varepsilon$.

Proof. In case of $\rho_{MT}(\delta_{x_1}, \delta_{x_1+\varepsilon})$ the supremum from Definition 7 is realized by

$$\psi = \mathbf{1}_{(-\infty, x_1]}(x) - \mathbf{1}_{(x_1, \infty)}(x).$$

In case of $\rho_{MT}(\delta_{x_1}, \delta_{x_1-\varepsilon})$ the supremum is realized by

$$\psi = (-1)\mathbf{1}_{(-\infty, x_1-2]}(x) + (x - x_1 + 1)\mathbf{1}_{(x_1-2, x_1)} + \mathbf{1}_{[x_1, \infty)}.$$

Note that we cannot use function $\psi = \mathbf{1}_{(-\infty, x_1)} - \mathbf{1}_{[x_1, \infty)}$, since it is not left-continuous in x_1 . □

In Table 2 we summarize the behaviour of metrics considered in this section in the vicinity of points x_i .

Metric	$\ \cdot\ $	ρ_{MT}	ρ_W	ρ_F
Distance of δ_{x_1} and $\delta_{x_1+\varepsilon}$	2	2	ε	ε
Distance of δ_{x_1} and $\delta_{x_1-\varepsilon}$	2	ε	ε	ε

Table 2: Perturbations of δ_{x_1} calculated in various metrics.

The measure-transmission metric can be thought of as halfway between $\|\cdot\|$ and ρ_F . Namely, it has properties of the flat metric to the left of x_i and of the norm distance to the right of x_i , which corresponds to an energy barrier at discrete states x_i .

3 Modified framework of cell differentiation

The framework for modelling cell differentiation processes, introduced in [25] and briefly presented in Section 1, is given by the following equations:

$$\partial_t \mu(t) + \partial_x (g_1(v(t)) \mathbf{1}_{x \neq x_i}(x) \mu(t)) = p(v(t), x) \mu(t), \quad (3.1)$$

$$g_1(v(t)) \frac{D\mu(t)}{D\mathcal{L}^1}(x_i^+) = c_i(v(t)) \int_{\{x_i\}} d\mu(t), \quad i = 0, \dots, N \quad (3.2)$$

$$\mu(0) = \mu_0, \quad (3.3)$$

where $t \in \mathbb{R}^+$ and $x \in \mathbb{R}$. $x_0 < x_1 < \dots < x_N$ is a finite collection of points in \mathbb{R} , $\mathbf{1}_{x \neq x_i}$ is equal 1 if $x \in (x_0, x_1) \cup (x_1, x_2) \cup \dots \cup (x_{N-1}, x_N)$ and 0 otherwise. $\frac{D\mu}{D\mathcal{L}^1}$ denotes the density of measure μ with respect to the one-dimensional Lebesgue measure and $v(t) := \int_{\{x_N\}} d\mu(t)$ denotes the mass of point x_N . The initial datum μ_0 is a Radon measure supported on the interval $[x_0, x_N]$. The assumptions on coefficients are following.

Assumptions 10 (see [25], Assumptions 3.2). (i) $g_1(v) \in Lip^b(\mathbb{R})$, and $g_1 > 0$,

(ii) $p = p(v(t), x) = p_1(v(t))p_2(x)$,

(iii) $p_1(v) \in Lip^b(\mathbb{R})$,

(iv) $p_2(x) \in \mathcal{B}^b(\mathbb{R})$, $p_2(x) = 0$ for $x \in \mathbb{R} \setminus [x_0, x_N]$ and p_2 restricted to (x_{i-1}, x_i) is Lipschitz continuous for every $i \in \{1, \dots, N\}$,

$$(v) \quad c_i = c_i(v) \in Lip^b(\mathbb{R}), \quad i = 0, 1, \dots, N,$$

$$(vi) \quad c_i \geq 0, \quad i = 0, 1, \dots, N,$$

$$(vii) \quad c_N = 0.$$

Above, $\mathcal{B}^b(\mathbb{R})$ stands for the space of bounded Borel functions on \mathbb{R} and $Lip^b(\mathbb{R})$ for the space of bounded Lipschitz functions on \mathbb{R} . The solutions are defined as follows.

Definition 11 (ρ -measure-transmission solution, see Definition 3.3 from [25]). *Let μ_0 be a Radon measure supported on $[x_0, x_N]$. A measure-valued function $\mu \in C([0, \infty), (\mathcal{M}, \rho))$ with $\int_{\{x_N\}} d\mu(t) \in BV_{loc}([0, \infty))$ is called a ρ -measure-transmission solution of problem (3.1)–(3.3), if*

i) *for every $\phi \in C_c^\infty([0, \infty) \times \mathbb{R})$*

$$\begin{aligned} & - \int_{\mathbb{R}^+} \int_{\mathbb{R}} \partial_t \phi(t, x) d\mu(t)(x) dt - \int_{\mathbb{R}^+} \int_{\mathbb{R}} g_1(v(t)) \mathbf{1}_{x \neq x_i}(x) \partial_x \phi(t, x) d\mu(t)(x) dt \\ & = \int_{\mathbb{R}^+} \int_{\mathbb{R}} p_1(v(t)) p_2(x) \phi(t, x) d\mu(t)(x) dt + \int_{\mathbb{R}} \phi(0, x) d\mu_0(x), \end{aligned} \quad (3.4)$$

ii) *for every $t^* > 0$ there exists $\varepsilon(t^*)$ such that for every $t > t^*$ measure $\mu(t)$ is absolutely continuous with respect to the Lebesgue measure \mathcal{L}^1 for $x \in (x_i, x_i + \varepsilon)$ and for \mathcal{L}^1 a.e. $t \in (0, \infty)$*

$$\lim_{x \rightarrow x_i^+} g_1(v(t)) \frac{D\mu(t)}{D\mathcal{L}^1}(x) = c_i(v(t)) \int_{x_i} d\mu(t),$$

iii) *for every $i = 0, 1, \dots, N$ we have $\int_{\{x_i\}} d\mu(t) \rightarrow \int_{\{x_i\}} d\mu(0)$ as $t \rightarrow 0$.*

Above, $BV_{loc}([0, \infty))$ is the space of right-continuous functions, which are of bounded variation on every finite subinterval of $[0, \infty)$ (we refer e.g. to [29] for definition and properties of BV functions).

The following theorem summarizes the analytical content of [25].

Theorem 12 (Existence and uniqueness of ρ_F -measure-transmission solutions). *For every Radon measure $\mu_0 \in \mathcal{M}(\mathbb{R})$ such that $\text{supp}(\mu_0) \subset [x_0, x_N]$, there exists a unique measure-transmission solution of problem (3.1)–(3.3) in the sense of Definition 11 with $\rho = \rho_F$.*

As observed in Example 1, in case $\rho = \rho_F$ the solutions lack continuity with respect to perturbation of the initial condition. The choice of metric $\rho = \rho_{MT}$ fixes this defect. The well-posedness results in the new setting are contained in Theorem 13 (existence and uniqueness) and Theorem 15 (stability).

Theorem 13 (Existence and uniqueness of ρ_{MT} -measure-transmission solutions). *For every Radon measure $\mu_0 \in \mathcal{M}(\mathbb{R})$ such that $\text{supp}(\mu_0) \subset [x_0, x_N]$, there exists a unique measure-transmission solution of problem (3.1)–(3.3) in the sense of Definition 11 with $\rho = \rho_{MT}$.*

Proof. Observe that $C([0, \infty), (\mathcal{M}, \rho_{MT})) \subset C([0, \infty), (\mathcal{M}, \rho_F))$. Thus, uniqueness follows from Theorem 12. Existence of solutions is a consequence of observation that the proof of Lemma 4.9 from [25] carries over with no change to the case of ρ_{MT} . Thus, solutions defined explicitly by formulas (17)–(21) in [25] belong not only to $Lip_{loc}([0, T], (\mathcal{M}, \rho_F))$ but also to $Lip_{loc}([0, T], (\mathcal{M}, \rho_{MT}))$ and hence to $C([0, \infty), (\mathcal{M}, \rho_F))$. Change of the time variable in [25, Definition 6.1] preserves this regularity. Thus, solutions constructed in [25] belong in fact to $C([0, \infty), (\mathcal{M}, \rho_{MT}))$, which concludes the proof. \square

Remark 14. i) It is possible to adopt a more general approach to existence and uniqueness of solutions based on the superposition solution technique, see [30].

ii) The assumptions of Definition 11 can be relaxed. This leads to additional technical difficulties and is fully treated in [30], see also Remark 16.

Now, we formulate our main result. Let

- $\sup(c) := \max_{i \in \{0, \dots, N\}} \sup_{v \in \mathbb{R}} |c_i(v)|$,
- $\sup(g_1) := \sup_{v \in \mathbb{R}} g_1(v)$,
- $\min(g_1) := \min_{k=1,2} \inf_{t \in [0, \infty)} g_1(v_k(t))$, where $v_k(t) = \int_{\{x_N\}} d\mu_k(t)$,
- $\text{Lip}(g_1)$ be the Lipschitz constant of g_1 ,
- $\text{Lip}(c) := \max_{i \in \{0, \dots, N\}} \text{Lip}(c_i)$, where $\text{Lip}(c_i)$ are the Lipschitz constants of functions c_i ,
- $TV(\mu) := \int_{\mathbb{R}} d\mu$ be the total variation of μ .

Then the following stability theorem holds.

Theorem 15 (Stability of ρ_{MT} -measure-transmission solutions in case $p = 0$). *Let $\mu_1(t)$ and $\mu_2(t)$ be two ρ_{MT} -measure-transmission solutions of system (3.1)-(3.3) with $p \equiv 0$, corresponding to initial conditions $\mu_1(0)$ and $\mu_2(0)$, respectively. There exist constants α, β , dependent only on $\sup(c)$, $\sup(g_1)$, $\min(g_1)$, $\text{Lip}(g_1)$, $\text{Lip}(c)$, $TV(\mu_1(0))$, $TV(\mu_2(0))$ such that*

$$\rho_{MT}(\mu_1(t), \mu_2(t)) \leq e^{\alpha \lceil \frac{t}{\beta} \rceil} \rho_{MT}(\mu_1(0), \mu_2(0)), \quad (3.5)$$

where $\lceil \frac{t}{\beta} \rceil$ is the smallest integer greater or equal $\frac{t}{\beta}$.¹

The proof of Theorem 15 is presented in Section 4. Note that, for simplicity, we consider only case $p = 0$, postponing the full result to further work.

4 Proof of the stability theorem in case $p = 0$

In this chapter we prove Theorem 15. We consider, namely, the system of equations

$$\partial_t \mu(t) + \partial_x(g_1(v(t)) \mathbf{1}_{x \neq x_i}(x) \mu(t)) = 0, \quad (4.1)$$

$$g_1(v(t)) \frac{D\mu(t)}{D\mathcal{L}^1}(x_i^+) = c_i(v(t)) \int_{\{x_i\}} d\mu(t), \quad i = 0, \dots, N \quad (4.2)$$

$$\mu(0) = \mu_0, \quad (4.3)$$

which is a simplification of system (3.1)-(3.3) obtained by taking $p = 0$.

To prove Theorem 15 we take two ρ_{MT} -measure-transmission solutions $\mu_1(t), \mu_2(t)$, denote $v_j(t) := \int_{\{x_N\}} d\mu_j(t)$ for $j \in \{1, 2\}$ and proceed in the following steps:

1. We prove a 'superposition principle' (see [31, 32]) for system (4.1)-(4.3), which allows us to express its solutions as certain combinations over characteristics called *superposition solutions*.
2. We obtain an estimate of $\int_0^T |v_1(t) - v_2(t)| dt$ in terms of $\rho_{MT}(\mu_1(0), \mu_2(0))$ and $\int_U d\mu_1(0), \int_U d\mu_2(0)$, where U is some neighborhood of x_N (Nonlinear Estimate).

¹In particular, $\lim_{t \rightarrow 0^+} e^{\alpha \lceil \frac{t}{\beta} \rceil} = e^\alpha$.

3. We obtain an estimate of $\rho_{MT}(\mu_1(t), \mu_2(t))$ for small t in terms of $\int_0^t |v_1(s) - v_2(s)| ds$ and $\rho_{MT}(\mu_1(0), \mu_2(0))$ (Linear Estimate).
4. We substitute the Nonlinear Estimate into the Linear Estimate to obtain an estimate of $\rho_{MT}(\mu_1(t), \mu_2(t))$ in terms of $\rho_{MT}(\mu_1(0), \mu_2(0))$ for small t .
5. We prolong the estimate to large t .

Remark 16. Steps 2-5, presented above, are based solely on the fact that every measure-transmission solution can be represented as superposition solution, i.e. in terms of formulas (4.8)-(4.9). Thus, estimate (3.5) holds true for every pair of measure-valued functions $\mu_1, \mu_2 : [0, T] \rightarrow \mathcal{M}(\mathbb{R})$, which satisfy (4.8)-(4.9). In particular, if the definition of measure-transmission solutions is modified in a way, which preserves the superposition principle, then stability estimate (3.5) remains valid. This comment is motivated by the fact that uniqueness criteria ii)-iii) of Definition 11, introduced in [25], which are an interpretation of the measure-transmission conditions (3.2), are somewhat artificial. More natural uniqueness criteria in the definition of solutions are studied in [30], where also, in contrast to [25], *detailed* proofs of existence and uniqueness of measure-transmission solutions are provided. As noted above, the stability estimate (3.5) carries over also to that case.

4.1 Superposition principle

In this section we show that measure-transmission solutions can be represented in terms of characteristics. Let, namely, T_{max} , G and $\tau(x_b)$, where $x_b \in \mathbb{R}$, be defined by

$$T_{max} := \frac{\min_{i \in \{1, 2, \dots, N\}} |x_i - x_{i-1}|}{\sup(g_1)}, \quad (4.4)$$

$$G(t) := \int_0^t g_1(v(s)) ds, \quad (4.5)$$

$$\tau(x_b) := \inf\{t \in [0, \infty) : x_b + G(t) \in \{x_0, x_1, \dots, x_N\}\}. \quad (4.6)$$

Let, moreover, $X(x_b, 0, r, \cdot)$ be, for $r \geq \tau(x_b)$, an absolutely continuous solution of equation $\dot{x} = \mathbf{1}_{x \neq x_i} g_1(v)$ given by formula

$$X(x_b, 0, r, t) := \begin{cases} x_b + G(t) & \text{for } t \leq \tau(x_b), \\ x_b + G(\tau(x_b)) & \text{for } \tau(x_b) < t \leq r, \\ x_b + G(\tau(x_b)) + G(t) - G(r) & \text{for } r < t \leq T. \end{cases} \quad (4.7)$$

We interpret $X(x_b, 0, r, \cdot)$ as the unique characteristic generated by $g_1(v)$ with a branching time r , see Figure 3. We obtain the following result.

Proposition 17 (Superposition principle). *Let μ be a ρ_{MT} -superposition solution of (4.1)-(4.3). Then for every bounded Borel function $\phi \in \mathcal{B}^b(\mathbb{R})$ and $T < T_{max}$ with T_{max} given by (4.4) we have*

$$\int_{\mathbb{R}} \phi d\mu(T) = \int_{\mathbb{R}} \left(\int_{[0, T]} \phi(X(x_b, 0, r, T)) d\eta_{x_b}(r) \right) d\mu(0)(x_b), \quad (4.8)$$

where

$$d\eta_{x_b}(r) := \begin{cases} e^{-\int_{\tau(x_b)}^r c_\lambda(v(s)) ds} \delta_T(dr) + c_\lambda(v(r)) e^{-\int_{\tau(x_b)}^r c_\lambda(v(s)) ds} \mathbf{1}_{[\tau(x_b), T]}(r) dr & \text{if } x_{\lambda-1} < x_b \leq x_\lambda \text{ for some } \lambda \in \{1, \dots, N-1\} \text{ and } \tau(x_b) \in [0, T], \\ \delta_T(dr) \text{ otherwise.} & \end{cases} \quad (4.9)$$

Remark 18. By the general superposition principle for continuity equation, see [32, Theorem 6.2.2], we obtain that there *exist* measures η_{x_b} such that (4.8) holds. Proposition 17 provides, in addition, an explicit formula for η_{x_b} , which is useful in subsequent computations.

Proof of Proposition 17. It is a simple calculation that η_{x_b} is a probability measure for every x_b . Thus, it remains to show that the left-hand side (LHS) of (4.8), calculated using formulas (17)-(20) and Definition 6.1 from [25], equals the right-hand side (RHS) of (4.8) calculated explicitly using formula (4.9). We proceed in two steps: $g_1 \equiv 1$ and arbitrary g_1 . In the following, for fixed solution μ , we denote $c_i(s) := c_i(v(s))$, $i \in \{0, 1, \dots, N\}$ and $g_1(s) := g_1(v(s))$. Functions h_i are defined by formula (18) from [25] and by 'characteristics end in A' we mean that $X(x_b, 0, r, T) \in A$.

Step 1 ($g_1 \equiv 1$). We begin with three special cases.

- (a) $\phi = \mathbf{1}_A$ with $A \subset (x_{i-1} + T, x_i)$ for some $i \in \{0, 1, \dots, N\}$. Then characteristics ending in A have the shape as in the left panel of Figure 3. We obtain

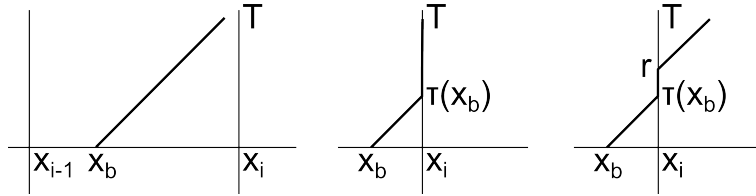


Figure 3: Three types of characteristics of system (4.1)-(4.3) for $g_1 \equiv 1$. In the first case $\tau(x_b) > T$ (left panel). In the second case, $0 \leq \tau(x_b) \leq T$ and $r \geq T$ (middle panel). In the last case $0 \leq \tau(x_b) \leq r < T$ (right panel).

$$\begin{aligned} RHS &= \int_{\mathbb{R}} \left(\int_{[0, T]} \mathbf{1}_A(X(x_b, 0, r, T)) \delta_T(dr) \right) d\mu(0)(x_b) = \int_{\mathbb{R}} \mathbf{1}_A(x_b + T) d\mu(0)(x_b) = \int_{A-T} d\mu(0)(x_b), \\ LHS &= \mu(T)(A) = \int_{A-T} d\mu(0)(x_b), \end{aligned}$$

where $A - T = \{x : x + T \in A\}$ and we used formula (19) from [25] to calculate LHS.

- (b) $\phi = \mathbf{1}_A$ with $A = \{x_i\}$ for some $i \in \{0, \dots, N\}$. Characteristics ending in A have the shape as in the middle panel of Figure 3. We obtain

$$\begin{aligned} RHS &= \int_{[x_i - T, x_i]} \left(\int_{[\tau(x_b), T]} \mathbf{1}_{\{x_i\}}(X(x_b, 0, r, T)) e^{-\int_{\tau(x_b)}^T c_i(s) ds} \delta_T(dr) \right) d\mu(0)(x_b) \\ &= \int_{[x_i - T, x_i]} \left(\mathbf{1}_{\{x_i\}}(X(x_b, 0, T, T)) e^{-\int_{\tau(x_b)}^T c_i(s) ds} \right) d\mu(0)(x_b), \\ LHS &= \mu(T)(\{x_i\}) = e^{-\int_0^T c_i(s) ds} \int_{\{x_i\}} d\mu(0) + \int_{(0, T]} h_i(dr) e^{-\int_r^T c_i(s) ds} \\ &= e^{-\int_0^T c_i(s) ds} \int_{\{x_i\}} d\mu(0) + \int_{[x_i - T, x_i]} e^{-\int_{\tau(x_b)}^T c_i(s) ds} d\mu(0)(x_b) \\ &= \int_{[x_i - T, x_i]} e^{-\int_{\tau(x_b)}^T c_i(s) ds} d\mu(0)(x_b), \end{aligned}$$

where we used formulas (18) and (20) from [25] to calculate LHS.

(c) $\phi = \mathbf{1}_A$ with $A \subset (x_i, x_i + T]$ for some $i \in \{0, 1, \dots, N\}$. Here, the characteristics assume the shape depicted in the right panel of Figure 3. As a result,

$$\begin{aligned}
RHS &= \int_{[x_i-T, x_i]} \left(\int_{[\tau(x_b), T]} \mathbf{1}_A(X(x_b, 0, r, T)) c_i(r) e^{-\int_{\tau(x_b)}^r c_i(s) ds} dr \right) d\mu(0)(x_b), \\
LHS &= \int_{T+x_i-A} f_i(r) dr = \int_{T+x_i-A} c_i(r) \left(\int_{\{x_i\}} d\mu(r)(x_b) \right) dr \\
&= \int_{T+x_i-A} c_i(r) \left[e^{-\int_0^r c_i(s) ds} \int_{\{x_i\}} d\mu(0) + \int_{(0, r]} h_i(d\tau) e^{-\int_{\tau}^r c_i(s) ds} \right] dr \\
&= \int_{[x_i-T, x_i]} \int_{\tau(x_b)}^T \mathbf{1}_A(x_i + T - r) e^{-\int_{\tau(x_b)}^r c_i(s) ds} c_i(r) dr d\mu(0)(x_b) \\
&= \int_{[x_i-T, x_i]} \int_{\tau(x_b)}^T c_i(r) e^{-\int_{\tau(x_b)}^r c_i(s) ds} \mathbf{1}_A(X(x_b, 0, r, T)) dr d\mu(0)(x_b),
\end{aligned}$$

where $T + x_i - A := \{t : x_i + T - t \in A\}$ and we used in turn formulas (19), (18), (20) from [25] as well as the Fubini theorem to compute LHS.

We observe that in every case $RHS = LHS$. Since functions of the form (a), (b), (c) generate the whole set of Borel-measurable functions on \mathbb{R} , we conclude.

Step 2 (arbitrary g_1). We use [25, Definition 6.1] and handle similarly as in the proof of [25, Theorem 6.2]. Namely, we define

$$\begin{aligned}
\tilde{t}(t) &:= \int_0^t g_1(s) ds, \quad d\tilde{t} := g_1(t) dt, \\
\tilde{c}_i(\tilde{t}) &:= \frac{c_i(t(\tilde{t}))}{g_1(t(\tilde{t}))}, \quad i = 0, 1, \dots, N, \\
\tilde{X}(x, 0, \tilde{r}(r), \tilde{s}(s)) &:= X(x, 0, r, s), \\
\tilde{\mu}(\tilde{t}(t)) &:= \mu(t).
\end{aligned}$$

Due to this transformation, $\tilde{\mu}$ satisfies equation (4.1) with velocity $\tilde{g}_1 \equiv 1$. Thus, using Step 1, we can write

$$\int_{\mathbb{R}} \phi d\mu(T) = \int_{\mathbb{R}} \phi d\tilde{\mu}(\tilde{T}) = \int_{\mathbb{R}} \left(\int_{[0, \tilde{T}]} \phi(\tilde{X}(x_b, 0, \tilde{r}, \tilde{T})) d\tilde{\eta}_{x_b}(\tilde{r}) \right) d\tilde{\mu}(0)(x_b).$$

Now, we transform the inner integral, using the change of variables defined above. There are two cases, depending on the value of parameter x_b .

- $\tilde{\eta}_{x_b} = \delta_{\tilde{T}}(d\tilde{r})$. Then

$$\int_{[0, \tilde{T}]} \phi(\tilde{X}(x_b, 0, \tilde{r}, \tilde{T})) d\tilde{\eta}_{x_b}(\tilde{r}) = \phi(X(x_b, 0, T, T)) = \int_{[0, T]} \phi(X(x_b, 0, r, T)) \delta_T(dr).$$

- $\tilde{\eta}_{x_b} = e^{-\int_{\tilde{\tau}(x_b)}^{\tilde{T}} \tilde{c}_i(\tilde{s}) d\tilde{s}} \delta_{\tilde{T}}(d\tilde{r}) + \tilde{c}_i(\tilde{r}) e^{-\int_{\tilde{\tau}(x_b)}^{\tilde{T}} \tilde{c}_i(\tilde{s}) d\tilde{s}} \mathbf{1}_{[\tilde{\tau}(x_b), \tilde{T}]}(\tilde{r}) d\tilde{r}$. Then

$$\begin{aligned}
& \int_{[0, \tilde{T}]} \phi(\tilde{X}(x_b, 0, \tilde{r}, \tilde{T})) d\tilde{\eta}_{x_b}(\tilde{r}) \\
&= \int_{[0, \tilde{T}]} \phi(X(x_b, 0, r, T)) \left[e^{-\int_{\tilde{\tau}(x_b)}^{\tilde{T}} \tilde{c}_i(\tilde{s}) d\tilde{s}} \delta_{\tilde{T}}(d\tilde{r}) + \tilde{c}_i(\tilde{r}) e^{-\int_{\tilde{\tau}(x_b)}^{\tilde{T}} \tilde{c}_i(\tilde{s}) d\tilde{s}} \mathbf{1}_{[\tilde{\tau}(x_b), \tilde{T}]}(\tilde{r}) d\tilde{r} \right] \\
&= \int_{[0, T]} \phi(X(x_b, 0, r, T)) \left[e^{-\int_{\tau(x_b)}^T c_i(s) ds} \delta_T(dr) + c_i(r) e^{-\int_{\tau(x_b)}^T c_i(s) ds} \mathbf{1}_{[\tau(x_b), T]}(r) dr \right].
\end{aligned}$$

Hence,

$$\begin{aligned}
\int_{\mathbb{R}} \phi d\mu(T) &= \int_{\mathbb{R}} \phi d\tilde{\mu}(\tilde{T}) = \int_{\mathbb{R}} \left(\int_{[0, \tilde{T}]} \phi(\tilde{X}(x_b, 0, \tilde{r}, \tilde{T})) d\tilde{\eta}_{x_b}(\tilde{r}) \right) d\tilde{\mu}(0)(x_b) \\
&= \int_{\mathbb{R}} \left(\int_{[0, T]} \phi(X(x_b, 0, r, T)) d\eta_{x_b}(r) \right) d\mu(0)(x_b).
\end{aligned}$$

□

Remark 19. A similar calculation, omitted here for simplicity, allows us to prove that for every ρ_{MT} -measure-transmission solution of system (3.1)-(3.3) and for every $\phi \in \mathcal{B}(\mathbb{R})$ and $t \in [0, T]$, $T < T_{max}$, we have

$$\int_{\mathbb{R}} \phi d\mu(t) = \int_{\mathbb{R}} \left(\int_{[0, T]} e^{\int_0^t p(s, X(x_b, 0, r, s)) ds} \phi(X(x_b, 0, r, t)) d\eta_{x_b}(r) \right) d\mu(0)(x_b), \quad (4.10)$$

where η_{x_b} is defined by (4.9).

4.2 Nonlinear estimate

Our goal here is to estimate $\int_0^T |v_1(t) - v_2(t)| dt$ in terms of $\rho_{MT}(\mu_1(0), \mu_2(0))$ where $T < T_{max}$ and T_{max} is given by (4.4). To this end, we observe that by Proposition 17 v_j can be expressed by

$$v_j(t) = \int_{[x_N - \int_0^t g_1(v_j(s)) ds, x_N]} d\mu_j(0) = \int_{[x_N - G_j(t), x_N]} d\mu_j(0), \quad (4.11)$$

where

$$G_j(t) := \int_0^t g_1(v_j(s)) ds, \quad (4.12)$$

and use the fact that for $p = 0$

$$\min(g_1) := \min_{j \in \{1, 2\}} \inf_{t \in [0, \infty)} g_1(v_j(t)) > 0 \quad (4.13)$$

due to boundedness of v_j and continuity as well as positivity of g_1 .

Denote $\min(G_j) := \min(G_1, G_2)$ and $\max(G_j) := \max(G_1, G_2)$. Using (4.11) we obtain

$$\begin{aligned}
\int_0^T |v_1(t) - v_2(t)| dt &= \int_0^T \left| \int_{[x_N - G_1(t), x_N]} d\mu_1(0) - \int_{[x_N - G_2(t), x_N]} d\mu_2(0) \right| dt \leq \\
\int_0^T \left| \int_{[x_N - \min(G_j), x_N]} d(\mu_1(0) - \mu_2(0)) \right| dt &+ \int_0^T \int_{[x_N - \max(G_j), x_N - \min(G_j))} d(\mu_1(0) + \mu_2(0)) dt = \\
&I_1 + I_2.
\end{aligned}$$

Let

$$\begin{aligned}
B &:= \left\{ t : \int_{[x_N - \min(G_j)(t), x_N]} d(\mu_1(0) - \mu_2(0)) \geq 0 \right\}, \\
\tau_j(x_b) &:= \sup\{t > 0 : x_b + G_j(t) < x_N\}, \\
\tau_{\min}(x_b) &:= \min(\tau_1(x_b), \tau_2(x_b)).
\end{aligned} \tag{4.14}$$

Then, by the Fubini theorem (see Figure 4), I_1 is equal to

$$\begin{aligned}
& \int_0^T (\mathbf{1}_B - \mathbf{1}_{\mathbb{R} \setminus B}) \int_{[x_N - \min(G_j)(t), x_N]} d(\mu_1(0) - \mu_2(0)) dt \\
&= \int_{[x_N - \min(G_j)(T), x_N]} \left(\int_{\tau_{\min}(x_b)}^T (\mathbf{1}_B - \mathbf{1}_{\mathbb{R} \setminus B})(t) dt \right) d(\mu_1(0) - \mu_2(0))(x_b) \\
&= \int_{\mathbb{R}} \chi(x_b) d(\mu_1(0) - \mu_2(0))(x_b).
\end{aligned}$$

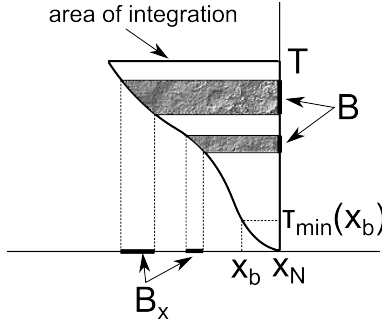


Figure 4: The area of integration in I_1 . The integral can be interpreted as a double integral of function $(\mathbf{1}_B - \mathbf{1}_{\mathbb{R} \setminus B})(t)$, which is positive in the shaded region and negative otherwise, with respect to the product measure $(\mu_1(0) - \mu_2(0)) \times dt$. B_x is the set of all x for which $\int_{[x, x_N]} d(\mu_1(0) - \mu_2(0))$ is nonnegative.

Function

$$\chi(x) := \begin{cases} \int_{\tau_{\min}(x)}^T (\mathbf{1}_B - \mathbf{1}_{\mathbb{R} \setminus B})(t) dt & \text{if } x \in [x_N - \min(G_j)(T), x_N], \\ 0 & \text{otherwise} \end{cases}$$

belongs to W_{MT}^b . Moreover,

$$|\chi(x)| \leq T$$

and

$$|\chi'(x)| \leq |(\mathbf{1}_B - \mathbf{1}_{\mathbb{R} \setminus B})| |\tau'_{\min}(x)| \leq \frac{1}{\min(g_1)},$$

where $\min(g_1) > 0$ by (4.13). As a consequence, $\chi = \max\left(\frac{1}{\min(g_1)}, T\right) \chi_1$, where χ_1 belongs to B_{MT} . This leads to conclusion that

$$I_1 \leq \max\left(\frac{1}{\min(g_1)}, T\right) \rho_{MT}(\mu_1(0), \mu_2(0)).$$

Denoting $J_{\max} = (x_N - \max(G_1(T), G_2(T)), x_N)$ and $J_{\min} = (x_N - \min(G_1(T), G_2(T)), x_N]$ and using the Fubini theorem as well as Proposition 35 we estimate I_2 by

$$\begin{aligned}
I_2 &\leq \sup_{x \in J_{\min}} |\tau_1(x) - \tau_2(x)| (\mu_1(0)(J_{\max}) + \mu_2(0)(J_{\max})) \\
&\leq (\mu_1(0)(J_{\max}) + \mu_2(0)(J_{\max})) \frac{\text{Lip}(g_1)}{\min(g_1)} \int_0^T |v_1(t) - v_2(t)| dt.
\end{aligned}$$

Combining estimates for I_1 and I_2 we obtain for T small enough

$$\int_0^T |v_1(t) - v_2(t)| dt \leq \max\left(\frac{1}{\min(g_1)}, T\right) \frac{1}{\left(1 - \frac{\text{Lip}(g_1)}{\min(g_1)}(\mu_1(0)(J_{\max}) + \mu_2(0)(J_{\max}))\right)} \rho_{MT}(\mu_1(0), \mu_2(0)). \quad (4.15)$$

Note that the maximum time T , up to which estimate (4.15) is valid, strongly depends on $\mu_1(0)$ and $\mu_2(0)$ via J_{\max} and cannot be controlled easily. Importantly, however, x_N does not belong to J_{\max} , which will allow us to prolong the stability estimate to arbitrary times, see Section 4.4.

Remark 20. For $g_1 \equiv 1$ estimate (4.15) turns into

$$\int_0^T |v_1(t) - v_2(t)| dt \leq \max(1, T) \rho_{MT}(\mu_1(0), \mu_2(0)). \quad (4.16)$$

4.3 Linear estimate

In this section we estimate the quantity

$$\rho_{MT}(\mu_1(T), \mu_2(T)) := \sup_{\psi \in B_{MT}} \int_{\mathbb{R}} \psi d(\mu_2(T) - \mu_1(T))$$

for $T < T_{\max}$. The main idea consists in splitting the integral $\int_{\mathbb{R}} \psi d(\mu_2(T) - \mu_1(T))$ into

- parts that can be bounded in terms of $\int_0^T |v_1(s) - v_2(s)| ds$ and
- parts, which add up to $\int_{\mathbb{R}} \psi^0 d(\mu_2(0) - \mu_1(0))$ for some function $\psi^0 \in W_{MT}^b$.

Then we bound both of them by $C_1(t) \rho_{MT}(\mu_1(0), \mu_2(0))$.

To achieve this goal, we fix $T < T_{\max}$, where T_{\max} is given by (4.4), and assume without loss of generality (compare Remark 24) that $G_1(t) \leq G_2(t)$ for $0 \leq t \leq T$, where G_1, G_2 are given by (4.12). By the superposition principle (Proposition 17) we have

$$\int_{\mathbb{R}} \psi d\mu_j(T) = \int_{\mathbb{R}} \left(\int_{[0, T]} \psi(X_j(x_b, 0, r, t)) d\eta_{x_b}^j(r) \right) d\mu_j(0)(x_b), \quad (4.17)$$

where:

- $j \in \{1, 2\}$ enumerates the two solutions,
- X_j is the characteristic generated by $g_1(v_j)$ (see (4.7)),
-

$$\tau_j(x_b) := \inf\{t \in [0, \infty) : x_b + G_j(t) \in \{x_0, x_1, \dots, x_N\}\}, \quad (4.18)$$

-

$$d\eta_{x_b}^j(r) := \begin{cases} e^{-\int_{\tau_j(x_b)}^T c_\lambda(v_j(s)) ds} \delta_T(dr) + c_\lambda(v_j(r)) e^{-\int_{\tau_j(x_b)}^r c_\lambda(v_j(s)) ds} \mathbf{1}_{[\tau_j(x_b), T]}(r) dr \\ \text{if } x_{\lambda-1} < x_b \leq x_\lambda \text{ for some } \lambda \in \{1, \dots, N-1\} \text{ and } \tau_j(x_b) \in [0, T], \\ \delta_T(dr) \text{ otherwise.} \end{cases}$$

Using this representation, we split the integral $\int_{\mathbb{R}} \psi d(\mu_2(T) - \mu_1(T))$ into three main components with respect to the starting point of characteristics, x_b :

- Characteristics starting in $(x_{i-1}, x_i - G_2(T))$ (Fig. 5) – terms I_i ,
- Characteristics starting in $[x_i - G_2(T), x_i - G_1(T))$ (Fig. 6) – terms T_i ,
- Characteristics starting in $[x_i - G_1(T), x_i]$ (Fig. 7) – terms D_i .

We obtain

$$\int_{\mathbb{R}} \psi d(\mu_2(T) - \mu_1(T)) = D_0 + (I_1 + T_1 + D_1) + (I_2 + T_2 + D_2) + \cdots + (I_N + T_N + D_N),$$

where

$$\begin{aligned} I_i &= \int_{(x_{i-1}, x_i - G_2(T))} (H_2(x_b) d\mu_2(x_b) - H_1(x_b) d\mu_1(x_b)), \\ T_i &= \int_{[x_i - G_2(T), x_i - G_1(T))} (H_2(x_b) d\mu_2(x_b) - H_1(x_b) d\mu_1(x_b)), \\ D_i &= \int_{[x_i - G_1(T), x_i]} (H_2(x_b) d\mu_2(x_b) - H_1(x_b) d\mu_1(x_b)) \end{aligned}$$

and we denoted

$$H_j(x_b) = \left(\int_{[0, T]} \psi(X_j(x_b, 0, r, t)) d\eta_{x_b}^j(r) \right) d\mu_j(0)(x_b).$$

Now, we estimate I_1 , T_1 and D_1 , the calculations for other terms being similar. In the estimates, we further group the characteristics in respect to the branching point r and the point reached by characteristic at time T . For convenience, as before, the fact that a characteristic reaches set A at time T will be shortly expressed as 'characteristic ends in A '.

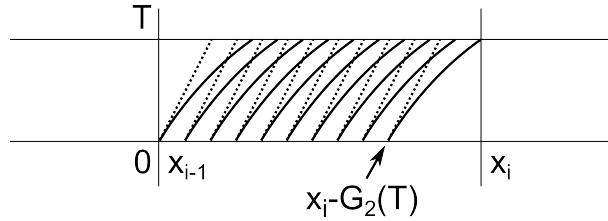


Figure 5: Characteristics generated by $g_1(v_1)$ (dotted) and $g_1(v_2)$ (solid) starting in $(x_{i-1}, x_i - G_2(T))$

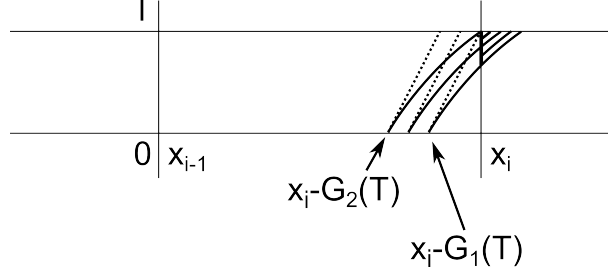


Figure 6: Characteristics generated by $g_1(v_1)$ (dotted) and $g_1(v_2)$ (solid) starting in $[x_i - G_2(T), x_i - G_1(T)]$. Characteristics corresponding to $g_1(v_2)$ arrive in x_i before time T and generate fans of characteristics whereas those corresponding to $g_1(v_1)$ do not.

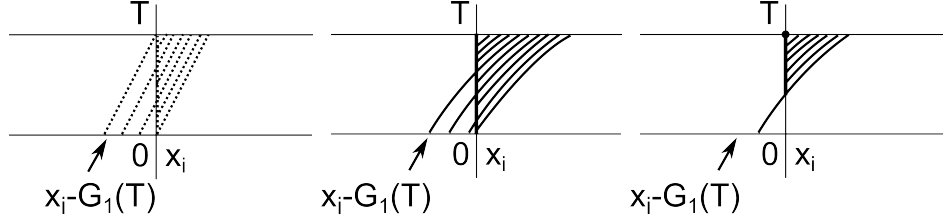


Figure 7: Characteristics generated by $g_1(v_1)$ (dotted, left panel) and $g_1(v_2)$ (solid, middle panel) starting in interval $[x_i - G_1(T), x_i]$. After arriving in x_i a given characteristic either spends an arbitrary period of time in x_i before leaving x_i or stays there until time T . Thus every characteristic coming to x_i branches generating a fan of characteristics (right panel).

Characteristics starting in $(x_0, x_1 - G_2(T))$

$$\begin{aligned}
 I_1 &= \int_{(x_0, x_1 - G_2(T))} [\psi(x_b + G_2(T)) d\mu_2(0)(x_b) - \psi(x_b + G_1(T)) d\mu_1(0)(x_b)] = \\
 &\quad \int_{(x_0, x_1 - G_2(T))} \psi(x_b + G_1(T)) d(\mu_2(0) - \mu_1(0)) + \\
 &\quad \int_{(x_0, x_1 - G_2(T))} [\psi(x_b + G_2(T)) - \psi(x_b + G_1(T))] d\mu_2(0) = U^{I_1} + V^{I_1}.
 \end{aligned}$$

Characteristics starting in $[x_1 - G_2(T), x_1 - G_1(T)]$

For μ_1 these characteristics do not branch before time T . In case of μ_2 , however, they reach x_1 before time T and therefore may branch. We obtain

$$\begin{aligned}
 T_1 &= \int_{[x_1 - G_2(T), x_1 - G_1(T))} \left\{ \psi(x_1) e^{-\int_{\tau_2(x_b)}^T c_1(v_2(s)) ds} d\mu_2(0) + \right. \\
 &\quad \left. \left(\int_{\tau_2(x_b)}^T e^{-\int_{\tau_2(x_b)}^r c_1(v_2(s)) ds} c_1(v_2(r)) \psi \left(x_1 + \int_r^T g_1(v_2(s)) ds \right) dr \right) d\mu_2(0) - \psi(x_b + G_1(T)) d\mu_1(0) \right\}.
 \end{aligned}$$

Consecutive terms in the integrand correspond to characteristics related to $\mu_2(0)$ ending in x_1 , related to $\mu_2(0)$ ending in (x_1, x_2) and related to $\mu_1(0)$. Further calculations lead to

$$\begin{aligned}
T_1 = & \int_{[x_1-G_2(T), x_1-G_1(T))} \psi(x_b + G_1(T)) d(\mu_2(0) - \mu_1(0)) + \\
& \int_{[x_1-G_2(T), x_1-G_1(T))} (\psi(x_1) - \psi(x_b + G_1(T))) d\mu_2(0) + \\
& \int_{[x_1-G_2(T), x_1-G_1(T))} \psi(x_1) \left(e^{-\int_{\tau_2(x_b)}^T c_1(v_2(s)) ds} - 1 \right) d\mu_2(0) + \\
& \int_{[x_1-G_2(T), x_1-G_1(T))} \left[\int_{\tau_2(x_b)}^T c_1(v_2(r)) e^{-\int_{\tau_2(x_b)}^r c_1(v_2(s)) ds} \psi \left(x_1 + \int_r^T g_1(v_2(s)) ds \right) dr \right] d\mu_2(0) = \\
& U^{T_1} + V_1^{T_1} + V_2^{T_1} + V_3^{T_1}.
\end{aligned}$$

Characteristics starting in $[x_1 - G_1(T), x_1]$

We subdivide those characteristics into three groups, see Fig. 8:

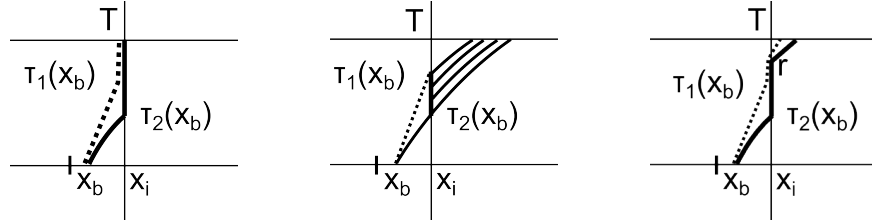


Figure 8: Sample characteristics starting in $x_b \in [x_i - G_1(T), x_i]$. Left panel. Characteristics starting in x_b and both ending in x_i . Middle panel. The fan of characteristics arriving at time $\tau_2(x_b)$ and leaving before $\tau_1(x_b)$ is small provided $|\tau_1(x_b) - \tau_2(x_b)|$ is small. Right panel. Characteristics starting in x_b and *both* branching off at time r .

- those ending in x_1 ,
- those ending in (x_1, x_2) and branching off between $\tau_2(x_b)$ and $\tau_1(x_b)$,
- those ending in (x_1, x_2) and branching off between $\tau_1(x_b)$ and T .

This leads to:

$$\begin{aligned}
D_1 = & \int_{[x_1-G_1(T), x_1]} \psi(x_1) e^{-\int_{\tau_2(x_b)}^T c_1(v_2(s)) ds} d\mu_2(0) - \int_{[x_1-G_1(T), x_1]} \psi(x_1) e^{-\int_{\tau_1(x_b)}^T c_1(v_1(s)) ds} d\mu_1(0) + \\
& \int_{[x_1-G_1(T), x_1]} \left(\int_{\tau_2(x_b)}^{\tau_1(x_b)} c_1(v_2(r)) e^{-\int_{\tau_2(x_b)}^r c_1(v_2(s)) ds} \psi \left(x_1 + \int_r^T g_1(v_2(s)) ds \right) dr \right) d\mu_2(0) + \\
& \int_{[x_1-G_1(T), x_1]} \int_{\tau_1(x_b)}^T \left[c_1(v_2(r)) e^{-\int_{\tau_2(x_b)}^r c_1(v_2(s)) ds} \psi \left(x_1 + \int_r^T g_1(v_2(s)) ds \right) dr d\mu_2(0) - \right. \\
& \left. c_1(v_1(r)) e^{-\int_{\tau_1(x_b)}^r c_1(v_1(s)) ds} \psi \left(x_1 + \int_r^T g_1(v_1(s)) ds \right) dr d\mu_1(0) \right] = \\
& 1^\circ + 2^\circ + 3^\circ.
\end{aligned}$$

Observe that

$$\begin{aligned}
1^\circ &= \psi(x_1) \int_{[x_1 - G_1(T), x_1]} e^{-\int_{\tau_1(x_b)}^T c_1(v_1(s)) ds} d(\mu_2(0) - \mu_1(0)) + \\
&\quad \psi(x_1) \int_{[x_1 - G_1(T), x_1]} \left(e^{-\int_{\tau_2(x_b)}^T c_1(v_2(s)) ds} - e^{-\int_{\tau_1(x_b)}^T c_1(v_1(s)) ds} \right) d\mu_2(0) = U_1^{D_1} + V_1^{D_1}, \\
2^\circ &= V_2^{D_1} \\
3^\circ &= \int_{[x_1 - G_1(T), x_1]} \left(\int_{\tau_1(x_b)}^T c_1(v_1(r)) e^{-\int_{\tau_1(x_b)}^r c_1(v_1(s)) ds} \psi \left(x_1 + \int_r^T g_1(v_1(s)) ds \right) dr \right) d(\mu_2(0) - \mu_1(0)) \\
&\quad + \int_{[x_1 - G_1(T), x_1]} \int_{\tau_1(x_b)}^T \left[c_1(v_2(r)) e^{-\int_{\tau_2(x_b)}^r c_1(v_2(s)) ds} \psi \left(x_1 + \int_r^T g_1(v_2(s)) ds \right) - \right. \\
&\quad \left. c_1(v_1(r)) e^{-\int_{\tau_1(x_b)}^r c_1(v_1(s)) ds} \psi \left(x_1 + \int_r^T g_1(v_1(s)) ds \right) \right] dr d\mu_2(0) = U_2^{D_1} + V_3^{D_1}.
\end{aligned}$$

Collecting similar terms we obtain

$$I_1 + T_1 + D_1 = (U^{I_1} + U^{T_1} + U^{D_1} + U^{D_2}) + (V^{I_1} + V_1^{T_1} + V_2^{T_1} + V_3^{T_1} + V_1^{D_1} + V_2^{D_1} + V_3^{D_1}).$$

Next, we estimate U-terms and V-terms using, mostly without explicit reference, Propositions 33-36.

U terms

$$(U^{I_1} + U^{T_1} + U_1^{D_1} + U_2^{D_1}) = \int_{(x_0, x_1]} \psi^0(x_b) d(\mu_2(0) - \mu_1(0))(x_b),$$

where

$$\psi^0(x_b) = \begin{cases} \psi(x_b + G_1(T)) & \text{for } x_0 < x_b < x_1 - G_1(T) \\ \psi(x_1) e^{-\int_{\tau_1(x_b)}^T c_1(v_1(s)) ds} + \\ \int_{\tau_1(x_b)}^T c_1(v_1(r)) e^{-\int_{\tau_1(x_b)}^r c_1(v_1(s)) ds} \psi \left(x_1 + \int_r^T g_1(v_1(s)) ds \right) dr & \text{for } x_1 - G_1(T) \leq x_b \leq x_1. \end{cases}$$

Note that ψ^0 is continuous in $x_1 - G_1(T)$ and left-continuous in x_1 . Let us compute explicitly the derivative of ψ^0 for $x_1 - G_1(T) < x_b < x_1$.

$$\begin{aligned}
(\psi^0)'(x_b) &= \tau_1'(x_b) c_1(v_1(\tau_1(x_b))) \psi(x_1) e^{-\int_{\tau_1(x_b)}^T c_1(v_1(s)) ds} \\
&\quad - \tau_1'(x_b) c_1(v_1(\tau_1(x_b))) \psi \left(x_1 + \int_{\tau_1(x_b)}^T g_1(v_1(s)) ds \right) \\
&\quad + \int_{\tau_1(x_b)}^T c_1(v_1(r)) e^{-\int_{\tau_1(x_b)}^r c_1(v_1(s)) ds} \tau_1'(x_b) c_1(v_1(\tau_1(x_b))) \psi \left(x_1 + \int_r^T g_1(v_1(s)) ds \right) dr.
\end{aligned}$$

Using $\tau_1'(x_b) \leq \frac{1}{\min(g_1)}$, which is bounded by (4.13), we arrive at

$$|(\psi^0)'(x_b)| \leq \frac{1}{\min(g_1)} (\sup(\psi) \sup(c_1) + \sup(\psi) \sup(c_1) + \sup(\psi) T (\sup(c_1))^2). \quad (4.19)$$

Similar calculations give analogous estimates for ψ^0 on $(x_{i-1}, x_i]$ for $i \in \{0, \dots, N\}$. Thus,

$$|\psi^0(x_b)| \leq \sup(\psi)$$

for all $x_b \in [x_0, x_N]$ and

$$\left| \frac{d}{dx_b} \psi^0(x_b) \right| \leq \max \left(\sup(\psi'), \frac{\sup(\psi) \sup(c)}{\min(g_1)} (2 + T \sup(c)) \right)$$

for $x_b \in (x_0, x_1) \cup (x_1, x_2) \cup \dots \cup (x_{N-1}, x_N)$, where $\sup(c) = \max_{i \in \{0,1,\dots,N\}} \sup(c_i)$.

V terms

$$|V^{I_1}| \leq \text{Lip}(\psi) \text{Lip}(g_1) \mu_2(0) ((x_0, x_1 - G_2(T)) \int_0^T |v_2(s) - v_1(s)| ds,$$

where we used $G_2(T) - G_1(T) \leq \text{Lip}(g_1) \int_0^T |v_2(s) - v_1(s)| ds$. Here and below $\text{Lip}(\psi)$ is the Lipschitz constant of ψ on interval $(x_{i-1}, x_i]$, which is bounded by 1.

$$\begin{aligned} |V_1^{T_1}| &\leq \text{Lip}(\psi) \text{Lip}(g_1) \mu_2(0) ([x_1 - G_2(T), x_1 - G_1(T)) \int_0^T |v_2(s) - v_1(s)| ds, \\ |V_2^{T_1}| &\leq |\psi(x_1)| \sup(c_1) \sup(T - \tau_2(x_b)) \mu_2(0) ([x_1 - G_2(T), x_1 - G_1(T)) \\ &\leq |\psi(x_1)| \sup(c_1) \frac{\text{Lip}(g_1)}{\min(g_1)} \mu_2(0) ([x_1 - G_2(T), x_1 - G_1(T)) \int_0^T |v_2(s) - v_1(s)| ds, \\ |V_3^{T_1}| &\leq \sup(\psi) \sup(c_1) \frac{\text{Lip}(g_1)}{\min(g_1)} \mu_2(0) ([x_1 - G_2(T), x_1 - G_1(T)) \int_0^T |v_2(s) - v_1(s)| ds, \\ |V_1^{D_1}| &\leq |\psi(x_1)| \left| \int_{\tau_2(x_b)}^T c_1(v_2(s)) ds - \int_{\tau_1(x_b)}^T c_1(v_1(s)) ds \right| \mu_2(0) ([x_1 - G_1(T), x_1]) \\ &\leq |\psi(x_1)| \left(\text{Lip}(c_1) \int_0^T |v_2(s) - v_1(s)| ds + \sup(c_1) \sup_{x_b} |\tau_2(x_b) - \tau_1(x_b)| \right) \mu_2(0) ([x_1 - G_1(T), x_1]) \\ &\leq |\psi(x_1)| \left(\text{Lip}(c_1) + \sup(c_1) \frac{\text{Lip}(g_1)}{\min(g_1)} \right) \mu_2(0) ([x_1 - G_1(T), x_1]) \int_0^T |v_2(s) - v_1(s)| ds, \\ |V_2^{D_1}| &\leq \sup_{x_b} |\tau_2(x_b) - \tau_1(x_b)| \sup(c_1) \sup(\psi) \mu_2(0) ([x_1 - G_1(T), x_1]) \\ &\leq \frac{\text{Lip}(g_1)}{\min(g_1)} \sup(c_1) \sup(\psi) \mu_2(0) ([x_1 - G_1(T), x_1]) \int_0^T |v_1(s) - v_2(s)| ds. \end{aligned}$$

To estimate $V_3^{D_1}$ let us first consider the inner integral

$$\begin{aligned} &\int_{\tau_1(x_b)}^T \left[c_1(v_2(r)) e^{-\int_{\tau_2(x_b)}^r c_1(v_2(s)) ds} \psi \left(x_1 + \int_r^T g_1(v_2(s)) ds \right) - \right. \\ &\quad \left. c_1(v_1(r)) e^{-\int_{\tau_1(x_b)}^r c_1(v_1(s)) ds} \psi \left(x_1 + \int_r^T g_1(v_1(s)) ds \right) \right] dr = \\ &\int_{\tau_1(x_b)}^T (c_1(v_2(r)) - c_1(v_1(r)) e^{-\int_{\tau_2(x_b)}^r c_1(v_2(s)) ds} \psi \left(x_1 + \int_r^T g_1(v_2(s)) ds \right) dr + \\ &\int_{\tau_1(x_b)}^T c_1(v_1(r)) \left(e^{-\int_{\tau_2(x_b)}^r c_1(v_2(s)) ds} - e^{-\int_{\tau_1(x_b)}^r c_1(v_1(s)) ds} \right) \psi \left(x_1 + \int_r^T g_1(v_2(s)) ds \right) dr + \\ &\int_{\tau_1(x_b)}^T c_1(v_1(r)) e^{-\int_{\tau_1(x_b)}^r c_1(v_1(s)) ds} \left(\psi \left(x_1 + \int_r^T g_1(v_2(s)) ds \right) - \psi \left(x_1 + \int_r^T g_1(v_1(s)) ds \right) \right) dr = \\ &\quad I_\alpha + I_\beta + I_\gamma. \end{aligned}$$

Now,

$$\begin{aligned}
|I_\alpha| &\leq \sup(\psi) \text{Lip}(c_1) \int_0^T |v_2(s) - v_1(s)| ds \\
|I_\beta| &\leq T \sup(\psi) \sup(c_1) \left(\sup(c_1) \frac{\text{Lip}(g_1)}{\min(g_1)} + \text{Lip}(c_1) \right) \int_0^T |v_2(s) - v_1(s)| ds \\
|I_\gamma| &\leq T \sup(c_1) \text{Lip}(\psi) \text{Lip}(g_1) \int_0^T |v_2(s) - v_1(s)| ds,
\end{aligned}$$

where for I_β we used the estimate

$$\begin{aligned}
\left| e^{-\int_{\tau_2}^r c_1(v_2(s)) ds} - e^{-\int_{\tau_1}^r c_1(v_1(s)) ds} \right| &\leq \left| \left(\int_{\tau_2}^r c_1(v_2(s)) ds - \int_{\tau_1}^r c_1(v_1(s)) ds \right) \right| \leq \\
&\sup(c_1) \sup |\tau_2 - \tau_1| + \text{Lip}(c_1) \int_0^T |v_2(s) - v_1(s)| ds \leq \\
&\left(\frac{\sup(c_1) \text{Lip}(g_1)}{\min(g_1)} + \text{Lip}(c_1) \right) \int_0^T |v_2(s) - v_1(s)| ds.
\end{aligned}$$

Thus,

$$\begin{aligned}
|V_3^{D_1}| &\leq (|I_\alpha| + |I_\beta| + |I_\gamma|) \mu_2(0)([x_1 - G_1(T), x_1]) \\
&\leq \left(\sup(\psi) \text{Lip}(c_1) + \sup(\psi) \sup(c_1) T \left(\sup(c_1) \frac{\text{Lip}(g_1)}{\min(g_1)} + \text{Lip}(c_1) \right) \right. \\
&\quad \left. + \text{Lip}(\psi) \sup(c_1) T \text{Lip}(g_1) \right) \mu_2(0)([x_1 - G_1(T), x_1]) \int_0^T |v_1(s) - v_2(s)| ds.
\end{aligned}$$

Combining U -terms and V -terms for $i \in \{0, \dots, N\}$ we obtain

$$\begin{aligned}
\left| \int_{\mathbb{R}} \psi d(\mu_2(T) - \mu_1(T)) \right| &\leq |D_0| + |I_1 + T_1 + D_1| + |I_2 + T_2 + D_2| + \dots + |I_N + T_N + D_N| \leq \\
&\int_{\mathbb{R}} \psi^0 d(\mu_2(0) - \mu_1(0)) + \\
&\left\{ \text{Lip}(\psi) \text{Lip}(g_1) + 2 \sup(\psi) \sup(c) \frac{\text{Lip}(g_1)}{\min(g_1)} + 2 \sup(\psi) \text{Lip}(c) + \right. \\
&\left. T \sup(c) \left[\sup(\psi) \left(\sup(c) \frac{\text{Lip}(g_1)}{\min(g_1)} + \text{Lip}(c) \right) + \text{Lip}(\psi) \text{Lip}(g_1) \right] \right\} TV(\mu_2(0)) \int_0^T |v_2(s) - v_1(s)| ds.
\end{aligned}$$

Above, $TV(\mu_2(0)) := \int_{\mathbb{R}} d\mu_2(0)$. Taking into account that $\|\psi^0\|_{W_{MT}^b} \leq \|\psi\|_{W_{MT}^b} \max\left(1, \frac{\sup(c)}{\min(g_1)}(2 + T \sup(c))\right)$ we obtain

$$\begin{aligned}
\rho_{MT}(\mu_1(T), \mu_2(T)) &= \sup_{\psi \in B_{MT}^{1,\infty}} \int_{\mathbb{R}} \psi d(\mu_2(T) - \mu_1(T)) \leq \\
&\max\left(1, \frac{\sup(c)}{\min(g_1)}(2 + T \sup(c))\right) \rho_{MT}(\mu_1(0), \mu_2(0)) + \\
&\left\{ \text{Lip}(g_1) + 2 \sup(c) \frac{\text{Lip}(g_1)}{\min(g_1)} + 2 \text{Lip}(c) + T \sup(c) \left[\sup(c) \frac{\text{Lip}(g_1)}{\min(g_1)} + \text{Lip}(c) + \text{Lip}(g_1) \right] \right\} \\
&TV(\mu_2(0)) \int_0^T |v_2(s) - v_1(s)| ds.
\end{aligned}$$

This in combination with (4.15) leads to the following local stability result.

Corollary 21 (Local in time stability estimate). For $0 < T < T_{max}$, where T_{max} is given by (4.4), we have

$$\rho_{MT}(\mu_1(T), \mu_2(T)) \leq C_1(T) \rho_{MT}(\mu_1(0), \mu_2(0)), \quad (4.20)$$

where

$$\begin{aligned} C_1(T) = & \max \left(1, \frac{\sup(c)}{\min(g_1)} (2 + T \sup(c)) \right) + \\ & \left\{ \text{Lip}(g_1) + 2 \sup(c) \frac{\text{Lip}(g_1)}{\min(g_1)} + 2 \text{Lip}(c) + T \sup(c) \left[\sup(c) \frac{\text{Lip}(g_1)}{\min(g_1)} + \text{Lip}(c) + \text{Lip}(g_1) \right] \right\} \times \\ & \times TV(\mu_2(0)) \max \left(\frac{1}{\min(g_1)}, T \right) \frac{1}{\left(1 - \frac{\text{Lip}(g_1)}{\min(g_1)} (\mu_1(0)(J_{\max}) + \mu_2(0)(J_{\max})) \right)}. \end{aligned} \quad (4.21)$$

The following two examples show that it is impossible to obtain a stability estimate with $C_1(0^+) = 1$ for arbitrary initial data.

Example 22. Take $\mu(0) = \delta_{x_1}$ and $\mu^\varepsilon(0) = \delta_{x_1 - \varepsilon}$ as well as $g_1 \equiv 1$ and c_1 constant. Then,

$$\rho_{MT}(\mu(0), \mu^\varepsilon(0)) = \varepsilon.$$

Let these measures evolve according to equation (4.1). For $t = \varepsilon$ we obtain

$$\begin{aligned} \mu^\varepsilon(t = \varepsilon) &= \delta_{x_1}, \\ \mu(t = \varepsilon) &= e^{-c_1 \varepsilon} \delta_{x_1} + c_1 e^{-c_1(\varepsilon - (x - x_1))} \mathbf{1}_{[x_1, x_1 + \varepsilon]}(x) \mathcal{L}^1(dx). \end{aligned}$$

Hence,

$$\rho_{MT}(\mu(\varepsilon), \mu^\varepsilon(\varepsilon)) = 2(1 - e^{-c_1 \varepsilon}) \simeq 2c_1 \varepsilon = 2c_1 \rho_{MT}(\mu(0), \mu^\varepsilon(0)).$$

We conclude that for every ε there exists a pair of measures $\mu_1(0)$ and $\mu_2(0)$ for which

$$\rho_{MT}(\mu_1(\varepsilon), \mu_2(\varepsilon)) \simeq 2c_1 \rho_{MT}(\mu_1(0), \mu_2(0)).$$

Example 23. Take two initial measures:

$$\mu(0) = \delta_{x_N} + \delta_y,$$

$$\mu^\varepsilon(0) = \delta_{x_N - \varepsilon} + \delta_y,$$

where $y \in (x_{N-1}, x_N)$ is such that $|x_N - y| > 1$. Clearly, $\rho_{MT}(\mu(0), \mu^\varepsilon(0)) = \varepsilon$. Take

$$g_1(v) = \begin{cases} \underline{g} & \text{for } v = 0, \\ 1 & \text{for } v = 1. \end{cases}$$

Let the measures evolve according to equation (4.1). For $\bar{t} = \frac{\varepsilon}{\underline{g}}$ we obtain $\mu(\bar{t}) = \delta_{x_N} + \delta_{y + \varepsilon/\underline{g}}$ and $\mu^\varepsilon(\bar{t}) = \delta_{x_N} + \delta_{y + \varepsilon}$. Thus,

$$\rho_{MT} \left(\mu \left(\frac{\varepsilon}{\underline{g}} \right), \mu^\varepsilon \left(\frac{\varepsilon}{\underline{g}} \right) \right) = \varepsilon \left(\frac{1}{\underline{g}} - 1 \right) = \left(\frac{1}{\underline{g}} - 1 \right) \rho_{MT}(\mu_1(0), \mu_2(0)).$$

Letting $\varepsilon \rightarrow 0$ leads us to conclusion that $C_1(0^+) \geq \left(\frac{1}{\underline{g}} - 1 \right)$.

Remark 24. It may happen that characteristics generated by $g_1(v_1)$ and $g_1(v_2)$ cross in such a way that although $G_1(T) \leq G_2(T)$ there exist certain x_b for which $\tau_1(x_b) < \tau_2(x_b)$ (see Fig. 9). The reader will easily modify the proof of the linear estimate to encompass such behavior.

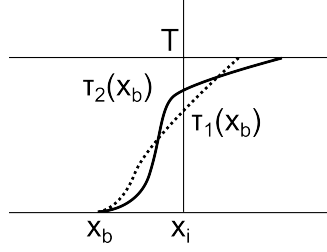


Figure 9: Crossing characteristics related to μ_1 (dotted) and μ_2 (solid). Although $\int_0^T g_1(v_1(s))ds < \int_0^T g_1(v_2(s))ds$, there exist certain x_b for which $\tau_1(x_b) < \tau_2(x_b)$.

4.4 Stability estimate for large times

Our goal is to obtain a global in time stability estimate with constant which depends only on the total mass of measures $\mu_1(0), \mu_2(0)$ and not on the initial mass distribution, i.e. the detailed structure of initial measures. We shall iterate estimate (4.20). Let, namely, $J_{max}^{t_0}$ be the interval J_{max} corresponding to initial time t_0 , i.e. $J_{max}^{t_0} := (x_N - \max(G_1(t_0, T), G_2(t_0, T)), x_N)$, where

$$G_j(t_0, T) := G_j(T) - G_j(t_0)$$

for $j = 1, 2$ and G_j are given by (4.12). Let, moreover,

$$\begin{aligned} C_1(t_0, T) := & \max \left(1, \frac{\sup(c)}{\min(g_1)} (2 + (T - t_0) \sup(c)) \right) + \\ & \left\{ \text{Lip}(g_1) + 2 \sup(c) \frac{\text{Lip}(g_1)}{\min(g_1)} + 2 \text{Lip}(c) + (T - t_0) \sup(c) \left[\sup(c) \frac{\text{Lip}(g_1)}{\min(g_1)} + \text{Lip}(c) + \text{Lip}(g_1) \right] \right\} \times \\ & \times TV(\mu_2(t_0)) \max \left(\frac{1}{\min(g_1)}, T - t_0 \right) \frac{1}{\left(1 - \frac{\text{Lip}(g_1)}{\min(g_1)} (\mu_1(t_0)(J_{max}^{t_0}) + \mu_2(t_0)(J_{max}^{t_0})) \right)} \end{aligned} \quad (4.22)$$

be a generalization of formula (4.21) to arbitrary initial times t_0 . We choose inductively the time points $0 = T_0 < T_1 < T_2 < \dots$ in such a way that for $j \in \{1, 2\}, k = 0, 1, \dots$

$$\mu_j(T_k)(J_{max}^{T_k}) \leq L := \frac{1}{4} \frac{\min(g_1)}{\text{Lip}(g_1)}, \quad (4.23)$$

$$\Delta T_k := T_{k+1} - T_k \leq \min \left(1, \min_{i \in \{0, \dots, N-1\}} \frac{|x_i - x_{i+1}|}{\sup(g_1)} \right) = \min(1, T_{max}), \quad (4.24)$$

and ΔT_k are maximal. To obtain the global estimate (3.5) we observe that

- $T_k \rightarrow \infty$ as $k \rightarrow \infty$,
- $\rho_{MT}(\mu_1(t), \mu_2(t)) \leq C_1(0, T_1) C_1(T_1, T_2) \times \dots \times C_1(T_{K-1}, T_K) \rho_{MT}(\mu_1(0), \mu_2(0))$ for $t \in [0, T_K]$, which follows by (4.20),
- constants $C_1(T_k, T_{k+1})$ can, by (4.22)-(4.24), be bounded in terms of a common constant κ , which implies

$$\rho_{MT}(\mu_1(t), \mu_2(t)) \leq \kappa^K \rho_{MT}(\mu_1(0), \mu_2(0)). \quad (4.25)$$

To finish the proof, we need to estimate, for every given $t > 0$, the 'number of iterations' K . The main difficulty lies in the fact that ΔT_k are not bounded away from 0. The first lemma, which is a consequence of (4.23)-(4.24), shows that if ΔT_k is small, then, informally speaking, the mass which is transported to x_N during the time interval $(T_k, T_{k+1}]$ has to be large.

Lemma 25. *Either*

$$\Delta T_k = \min(1, T_{max})$$

or

$$\mu_1(T_k)(J_{max}^{T_k, lc}) + \mu_2(T_k)(J_{max}^{T_k, lc}) \geq L,$$

where lc stands for left closure of an interval, i.e. $(a, b)^{lc} = [a, b)$.

Proof. If $\Delta T_k < \min(1, T_{max})$ then either

$$\mu_1(T_k)(J_{max}^{T_k}) = L$$

or

$$\mu_2(T_k)(J_{max}^{T_k}) = L$$

or both

$$\mu_1(T_k)(J_{max}^{T_k}) < L \text{ and } \mu_2(T_k)(J_{max}^{T_k}) < L.$$

In the latter case either $\mu_1(T_k)(J_{max}^{T_k, lc}) > L$ or $\mu_2(T_k)(J_{max}^{T_k, lc}) > L$ due to the fact that ΔT_k is the maximum time interval for which (4.23) holds. \square

Using Lemma 25, we estimate the number It_1 of iterations which are necessary for the whole mass from interval $(x_{N-1}, x_N]$ to 'be transported to x_N '.

Lemma 26. *Let $T_{intmin} := \frac{|x_N - x_{N-1}|}{\min(g_1)}$ be the maximum time necessary for all characteristics starting from interval $(x_{N-1}, x_N]$ to arrive in x_N . Then for*

$$k \geq It_1 := \left\lceil \frac{T_{intmin}}{\min(1, T_{max})} \right\rceil + \left\lceil \frac{\mu_1(0)((x_{N-1}, x_N)) + \mu_2(0)((x_{N-1}, x_N))}{L} \right\rceil + 1 \quad (4.26)$$

we have $\max(G_1(0, T_k), G_2(0, T_k)) \geq x_N - x_{N-1}$.

Proof. Suppose that $\max(G_1(0, T_k), G_2(0, T_k)) < x_N - x_{N-1}$. By formula (4.17)

$$\mu_j(T_{k-1})([x_N - G_j(T_{k-1}, T_k), x_N]) = \mu_j(0)([x_N - G_j(0, T_k), x_N - G_j(0, T_{k-1})]).$$

Furthermore, by Lemma 25, for every $l \in \{0, 1, \dots, k\}$ either

a) $\Delta T_l = \min(1, T_{max})$, where T_{max} is given by (4.4)

or

b) $\mu_1(0)([x_N - G_1(0, T_l), x_N - G_1(0, T_{l-1})]) + \mu_2(0)([x_N - G_2(0, T_l), x_N - G_2(0, T_{l-1})]) \geq L$.

Let

- $K_1 = \{l \in \{1, \dots, k\} : \text{a) holds}\}$ and
- $K_2 = \{l \in \{1, \dots, k\} : \text{b) holds}\}.$

By (4.26) for $k \geq \lceil It_1 \rceil$ either

$$\#(K_1) > \left\lceil \frac{T_{intmin}}{\min(1, T_{max})} \right\rceil$$

or

$$\#(K_2) > \left\lceil \frac{\mu_1(0)((x_{N-1}, x_N)) + \mu_2(0)((x_{N-1}, x_N))}{L} \right\rceil,$$

where $\#(K_1), \#(K_2)$ are the numbers of elements of K_1 and K_2 , respectively. This means that either

$$\begin{aligned} & \mu_1(0)((x_{N-1}, x_N)) + \mu_2(0)((x_{N-1}, x_N)) \geq \\ & \sum_{k \in K_1} \mu_1(0)([x_N - G_1(0, T_k), x_N - G_1(0, T_{k-1})]) + \mu_2(0)([x_N - G_1(0, T_k), x_N - G_1(0, T_{k-1})]) > \\ & L \left\lceil \frac{\mu_1(0)((x_{N-1}, x_N)) + \mu_2(0)((x_{N-1}, x_N))}{L} \right\rceil \geq \mu_1(0)((x_{N-1}, x_N)) + \mu_2(0)((x_{N-1}, x_N)) \end{aligned}$$

or

$$T_{intmin} \geq \sum_{k \in K_2} \Delta T_k > \left\lceil \frac{T_{intmin}}{\min(1, T_{max})} \right\rceil \min(1, T_{max}) \geq T_{intmin}.$$

In both cases we obtain contradiction. \square

Now, using the fact that $TV(\mu_j(t)) = TV(\mu_j(0))$ for all $t > 0$ by (4.8), we obtain that for $k_0 \geq 0$ and every

$$k \geq It_2 := \left\lceil \frac{T_{intmin}}{\min(1, T_{max})} \right\rceil + \left\lceil \frac{TV(\mu_1(0)) + TV(\mu_2(0))}{L} \right\rceil + 2$$

we have

$$\max(G_1(T_{k_0}, T_{k_0+k}), G_2(T_{k_0}, T_{k_0+k})) > x_N - x_{N-1}, \quad (4.27)$$

which follows by Lemma 26 applied for initial time T_{k_0} . On the other hand, for $j = 1, 2$

$$G_j(T_{k_0}, T_{k_0} + T_{int}) \leq x_N - x_{N-1}, \quad (4.28)$$

where $T_{int} := |x_{N-1} - x_N|/\sup(g_1)$ and we used the definition of G_j . Comparing (4.27) and (4.28) we obtain that for every $k_0 \geq 0$

$$T_{k_0} + T_{int} \leq T_{k_0 + It_2}. \quad (4.29)$$

Hence, iterating (4.29), we obtain $T_{k_0} + t \leq T_{k_0 + It_2 \lceil t/T_{int} \rceil}$ and, in particular,

$$t \leq T_{It_2 \lceil \frac{t}{T_{int}} \rceil} = T_K.$$

This, by (4.25), leads to the following conclusion.

Corollary 27.

$$\rho_{MT}(\mu_1(t), \mu_2(t)) \leq \kappa^{(It_2 \lceil \frac{t}{T_{int}} \rceil)} \rho_{MT}(\mu_1(0), \mu_2(0)), \quad (4.30)$$

where

$$\begin{aligned} T_{int} &= \frac{|x_N - x_{N-1}|}{\sup(g_1)}, \\ It_2 &= \left\lceil \frac{x_N - x_{N-1}}{\min(g_1) \min\left(1, \min_{i \in \{0, \dots, N-1\}} \frac{|x_i - x_{i+1}|}{\sup(g_1)}\right)} \right\rceil + \left\lceil \frac{TV(\mu_1(0)) + TV(\mu_2(0))}{L} \right\rceil + 2, \\ L &= \frac{1}{4} \frac{\min(g_1)}{\text{Lip}(g_1)}, \end{aligned}$$

$$\begin{aligned} \kappa &= \max \left(1, \frac{\sup(c)}{\min(g_1)} (2 + \sup(c)) \right) + \\ &2 \left\{ \text{Lip}(g_1) + 2 \sup(c) \frac{\text{Lip}(g_1)}{\min(g_1)} + 2 \text{Lip}(c) + \sup(c) \left[\sup(c) \frac{\text{Lip}(g_1)}{\min(g_1)} + \text{Lip}(c) + \text{Lip}(g_1) \right] \right\} \times \\ &\times TV(\mu_2(0)) \max \left(\frac{1}{\min(g_1)}, 1 \right). \end{aligned}$$

This proves Theorem 15.

Corollary 28. For $T < T_{int}$ we have

$$\int_0^T |v_1(t) - v_2(t)| dt \leq 2 \max \left(\frac{1}{\min(g_1)}, T_{int} \right) (It_1) \kappa^{It_2} \rho_{MT}(\mu_1(0), \mu_2(0)).$$

Proof. By (4.15) and (4.30) we obtain

$$\begin{aligned} \int_{T_k}^{T_{k+1}} |v_1(t) - v_2(t)| dt &\leq 2 \max \left(\frac{1}{\min(g_1)}, T_{int} \right) \rho_{MT}(\mu_1(T_k), \mu_2(T_k)) \\ &\leq 2 \max \left(\frac{1}{\min(g_1)}, T_{int} \right) \kappa^{It_2} \rho_{MT}(\mu_1(0), \mu_2(0)). \end{aligned}$$

Summing from $k = 0$ to $k = N - 1$, we conclude. \square

Remark 29. Time steps in iterations which lead to the global stability estimate (4.30) are *different* for every pair of initial measures. This is due to the fact that it is constant 'mass step' that is used rather than constant time step (see Lemma 25). In the end, however, the estimate has the same form for *every* pair of initial measures and depends only on their total variations. This is due to the fact that there is a finite potential for small time steps which depends only on the total variation of measures (see Lemma 26).

Remark 30. The standard theory of Lipschitz semiflows, see e.g. [33], does not allow us to obtain a global stability estimate from the local one. We recall that a mapping $\Phi : [0, \epsilon] \times [0, T] \times (M, \rho) \rightarrow (M, \rho)$ is called *Lipschitz semiflow* in metric space (M, ρ) if

- $\Phi(0, t, \mu) = \mu$ for $t \in [0, T]$,
- for t, s_1, s_2 such that $s_1, s_2, s_1 + s_2 \in [0, \epsilon]$ and $t, t + s_1 + s_2 \in [0, T]$ we have

$$\Phi(s_2, t + s_1, \Phi(s_1, t, \mu)) = \Phi(s_1 + s_2, t, \mu)$$

(semigroup property),

- for t, s_1, s_2 such that $s_1, s_2 \in [0, \epsilon]$ and $t, t + s_1, t + s_2$ belong to $[0, T]$ we have

$$\rho(\Phi(s_1, t, \mu_1), \Phi(s_2, t, \mu_2)) \leq L(\rho(\mu_1, \mu_2) + |s_1 - s_2|) \quad (4.31)$$

(Lipschitz continuity).

In our case, defining

$$\Phi(s, t, \mu) := \nu(s), \quad (4.32)$$

where $\nu(s)$ is the unique solution of problem (3.1)-(3.3) with initial condition $\nu(0) = \mu$, we would obtain a semiflow Φ , which would however not be Lipschitz due to the fact that the constant C_1 in estimate (4.21) cannot be chosen uniformly with respect to μ . Our elementary method of prolongation of the estimate overcomes this difficulty.

Remark 31. Stability of measure-transmission solutions of system (3.1)-(3.3) with respect to perturbation of the initial condition for general p remains open.

Remark 32. The stability result is important from the modelling point of view, since every reasonable model of reality needs to be stable. Moreover, the proof of Theorem 15 gives some hints, in the case $p = 0$, for construction of a convergent numerical scheme for simulating system (3.1)-(3.3). Such a scheme, based on particle methods, will be presented in a forthcoming paper.

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Appendix. Auxiliary estimates.

In this section we gather estimates, which for clarity of the exposition have been omitted from the main text.

Proposition 33. *For $x, y \geq 0$ it holds*

- i) $|e^x - 1| \leq |x|e^x$,
- ii) $|e^x - e^y| \leq |x - y|e^{\max(x,y)}$,
- iii) $|e^{-x} - e^{-y}| \leq |x - y|e^{-\min(x,y)}$.

Proof. Proof is elementary. □

Proposition 34. *Let $\xi(t)$ be an arbitrary non-negative Borel function on $[0, T]$. Then*

$$\sup_{t \in [0, T]} |e^{\xi(t)} - 1| \leq \sup_{t \in [0, T]} |\xi(t)| e^{\sup_{t \in [0, T]} |\xi(t)|}.$$

Proof.

$$\sup_{t \in [0, T]} |e^{\xi(t)} - 1| = \sup_{t \in [0, T]} \left| \int_0^{\xi(t)} e^s ds \right| \leq \sup_{t \in [0, T]} |\xi(t)| e^{\sup_{t \in [0, T]} |\xi(t)|}.$$

□

Proposition 35. *Let τ_1, τ_2 be defined by (4.14). For $0 < T < T_{max}$ we have*

$$|\tau_2(x_b) - \tau_1(x_b)| \leq \frac{\text{Lip}(g_1)}{\min(g_1)} \int_0^T |v_2(s) - v_1(s)| ds.$$

Proof.

$$\begin{aligned} |\tau_2(x_b) - \tau_1(x_b)| &\leq \frac{|(x_b + G_2(\min(\tau_1, \tau_2))) - (x_b + G_1(\min(\tau_1, \tau_2)))|}{\min(g_1)} \leq \\ &\frac{1}{\min(g_1)} \int_0^{\min(\tau_1, \tau_2)} |g_1(v_2(s)) - g_1(v_1(s))| ds \leq \frac{\text{Lip}(g_1)}{\min(g_1)} \int_0^T |v_2(s) - v_1(s)| ds. \end{aligned}$$

□

Proposition 36. *Let f^1, f^2 be two bounded functions defined on interval $[0, T]$. Denote $\sup(f) := \max(\sup(|f^1|), \sup(|f^2|))$. Then*

$$\left| e^{\int_r^T f^1(s)ds} - e^{\int_r^T f^2(s)ds} \right| \leq e^{3T \sup(f)} \int_r^T |f^2(s) - f^1(s)| ds.$$

Proof.

$$\begin{aligned} \left| e^{\int_r^T f^1(s)ds} - e^{\int_r^T f^2(s)ds} \right| &= \left| e^{\int_r^T f^1(s)ds} \left(1 - e^{\int_r^T (f^2(s) - f^1(s))ds} \right) \right| \leq \\ &e^{T \sup(|f^1|)} \left(\int_r^T |f^2(s) - f^1(s)| ds \right) e^{T(\sup(|f^1|) + \sup(|f^2|))} \leq \\ &e^{T(2 \sup(|f^1|) + \sup(|f^2|))} \int_r^T |f^2(s) - f^1(s)| ds. \end{aligned}$$

□

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